Last update: 10:57pm, Sept 7.

Math 564: Adv. Analysis 1 HOMEWORK 1 **Due: Sep 14, 11:59pm**

1. Let (X, d) be a metric space. Prove:

(a) Separability is hereditary for metric spaces, i.e. if X is separable, then every subspace $Y \subseteq X$ is also separable.

CAUTION: This is not true for general topological spaces. Think of an example.

(b) For any $Y \subseteq X$, its closure $\overline{Y}$ is equal to $\bigcap_{n \geqslant 1} B_{1/n}(Y)$, where

$$B_r(Y) := \{x \in X : d(x, Y) < r\}$$

and $d(x, Y) := \inf_{y \in Y} d(x, y)$. Conclude that every closed set is G_δ[1]; equivalently, every open set is F_σ.

2. Let A be a nonempty set (an alphabet) and consider the space $A^{\mathbb{N}}$ of infinite A-valued sequences, equipped with the metric d defined in class.

(a) [*Optional*] Prove that d is in fact an **ultrametric**, i.e. $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$ for each $x, y, z \in A^{\mathbb{N}}$.

(b) Prove that the metric space $(A^{\mathbb{N}}, d)$ is complete.

(c) Prove that $A^{\mathbb{N}}$ is compact if and only if A is finite. Please use the open covers definition of compactness here. (If you'd like a hint, please ask me.)

3. (a) Observe that in every metric space, the clopen sets form an algebra.

(b) Prove that in $2^{\mathbb{N}}$, the clopen sets are exactly the finite disjoint unions of cylinders.

4. Let X be a set and $\mathcal{C} \subseteq \mathscr{P}(X)$. Prove:

(a) $\langle \mathcal{C} \rangle = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$, where $\mathcal{C}_0 := \mathcal{C}$ and

$$\mathcal{C}_{n+1} := \{\text{complements and finite unions of sets in } \mathcal{C}_n\}.$$

(b) [*Optional*] $\langle \mathcal{C} \rangle_\sigma = \bigcup_{\alpha \in \omega_1} \mathcal{C}_\alpha$, where $\mathcal{C}_0 := \mathcal{C}$ and for $\alpha > 0$,

$$\mathcal{C}_\alpha := \left\{\text{complements and finite unions of sets in } \textstyle\bigcup_{\beta < \alpha} \mathcal{C}_\beta\right\}.$$

5. Let X be a set and $\mathcal{C} \subseteq \mathscr{P}(X)$. Put $\neg\mathcal{C} := \{S^c \in \mathcal{C} : S \in \mathcal{C}\}$. Let $\mathcal{S} \subseteq \mathscr{P}(X)$ be the smallest collection of sets containing $\mathcal{C} \cup \neg\mathcal{C}$ and closed under countable unions and countable intersections. Prove that $\mathcal{S} = \langle \mathcal{C} \rangle_\sigma$.

HINT: To show $\mathcal{S} \supseteq \langle \mathcal{C} \rangle_\sigma$, we do something counter-intuitive: we define an even smaller collection $\mathcal{S}' := \{S \in \mathcal{S} : S \text{ and } S^c \text{ are in } \mathcal{S}\}$ and show that $\mathcal{S}'$ is already a σ-algebra containing $\mathcal{C}$.

[1] A set is G_δ (resp. F_σ) if it is a countable intersection (resp. countable union) of open (resp. closed) sets.

6. The **Borel σ-algebra** of a metric (or topological) space X is the σ-algebra generated by the open sets of X. This is denoted by $\mathcal{B}(X)$ and the sets in it are called **Borel sets**.

   (a) Prove that if $\mathcal{U}$ is a countable basis for the topology of X [2], then $\mathcal{B}(X) = \langle \mathcal{U} \rangle_\sigma$.
   Remark: In fact, one can show that in a second countable topological space, every basis contains a countable subcollection that is still a basis, whence *every* basis generates the Borel σ-algebra.

   (b) Prove that the following collections generate the Borel σ-algebra of $\mathbb{R}^d$:

      (i) Balls with rational centers (i.e. in $\mathbb{Q}^d$) and rational radius.

      (ii) Open boxes.

      (iii) Boxes.

7. Prove Claim (b) for boxes in $\mathbb{R}^d$, namely: Let $\mathcal{A}$ denote the algebra of all finite unions of boxes in $\mathbb{R}^d$. For any finite partitions $\mathcal{P}$ and $\mathcal{Q}$ of a set $A \in \mathcal{A}$ into boxes, we have

$$\sum_{P \in \mathcal{P}} \tilde{\lambda}(Q) = \sum_{Q \in \mathcal{Q}} \tilde{\lambda}(P).$$

8. Finish the proof of countable additivity of λ on the algebra $\mathcal{A}$ of all finite unions of boxes in $\mathbb{R}^d$. More precisely:

   (i) Prove that $\lambda(B) = \sum_{n \in \mathbb{N}} \lambda(B_n)$ for an unbounded box $B \subseteq \mathbb{R}^d$ and a partition $\{B_n\}_{n \in \mathbb{N}}$ of B into boxes.

   Caution: An unbounded box has measure ∞ or 0.

   (ii) Finally conclude that $\lambda(A) = \sum_{n \in \mathbb{N}} \lambda(A_n)$ for a set $A \in \mathcal{A}$ and a partition $\{A_n\}_{n \in \mathbb{N}}$ of A into sets in $\mathcal{A}$.

9. Let X be a set and $\mathcal{A} \subseteq \mathscr{P}(X)$ be a collection of sets containing $\emptyset$ and covering X. Let $m : \mathcal{A} \to [0, \infty]$. Prove that the induced outer measure $m^* : \mathscr{P}(X) \to [0, \infty]$ is

   (a) monotone: $A \subseteq B$ implies $m^*(A) \leqslant m^*(B)$ for all $A, B \in \mathscr{P}(X)$.

   (b) (a bit more than) countably subadditive: $m^*(\bigcup_{n \in \mathbb{N}} B_n) \leqslant \sum_{n \in \mathbb{N}} m^*(B_n)$ for all $B_0, B_1, \ldots \in \mathscr{P}(X)$.

[2] This means that every open set in X is a union of sets in $\mathcal{U}$.